

CONTROLLABILITY OF FRACTIONAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION WITH INFINITE DELAY

EL HASSAN LAKHEL

Cadi Ayyad University, National School of Applied Sciences, 46000 Safi, Morocco

Abstract. In this paper we study the controllability of fractional neutral stochastic functional differential equations with infinite delay driven by fractional Brownian motion in a real separable Hilbert space. The controllability results are obtained by using stochastic analysis and a fixed-point strategy. Finally, an illustrative example is provided to demonstrate the effectiveness of the theoretical result.

Keywords: Controllability, fractional neutral functional differential equations, fractional powers of closed operators, infinite delay, fractional Brownian motion.

AMS Subject Classification: 35R10, 93B05 60G22, 60H20.

1. INTRODUCTION

Fractional Brownian motion (fBm) $\{B^H(t) : t \in \mathbb{R}\}$ is a Gaussian stochastic process, which depends on a parameter $H \in (0, 1)$ called Hurst index, for additional details on the fractional Brownian motion, we refer the reader to [20]. This stochastic process has self-similarity, stationary increments, and long-range dependence properties. It is known that fractional Brownian motion is a generalization of Brownian motion and it reduces to a standard Brownian motion when $H = \frac{1}{2}$. Fractional Brownian motion is not a semimartingale if $H \neq \frac{1}{2}$ (see Biagini *al.* [3]), the classical Itô theory cannot be used to construct a stochastic calculus with respect to fBm.

Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of physics, finance, electrical engineering, telecommunication networks, and so on. There has been a significant development in fractional differential equations. Some authors have considered fractional stochastic equations, we refer to Ahmed [1], El-Bori [10], Cui and Yan [8], Sakthivel et al. [25, 26]. The perturbed terms of these fractional equations are Wiener processes. For more details, one can see the monographs of Kilbas et al. [11], Zhou [28], and Zhou et al. [29] and the references therein.

In many areas of science, there has been an increasing interest in the investigation of the systems incorporating memory or aftereffect, i.e., there is the effect of delay on state equations. Therefore, there is a real need to discuss stochastic evolution systems with delay. In many mathematical models the claims often display long-range memories, possibly due to extreme weather, natural disasters, in some cases, many stochastic dynamical systems

¹Lakhel E.: e.lakhel@uca.ma (Corresponding author)

Moreover, control theory is an area of application-oriented mathematics which deals with basic principles underlying the analysis and design of control systems. Roughly speaking, controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Controllability plays a crucial role in a lot of control problems, such as the case of stabilization of unstable systems by feedback or optimal control [12, 13]. The controllability concept has been studied extensively in the fields of finite-dimensional systems, infinite-dimensional systems, hybrid systems, and behavioral systems. If a system cannot be controlled completely then different types of controllability can be defined such as approximate, null, local null and local approximate null controllability. For more details the reader may refer to [13, 23, 24] and the references therein. In this paper, we study the controllability of fractional neutral functional stochastic differential equations of the form

where $\frac{1}{2} < \alpha < 1$, $J^{1-\alpha}$ is the $(1-\alpha)$ -order Riemann-Liouville fractional integral operator, A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t \geq 0}$, in a Hilbert space X ; B^H is a fractional Brownian motion with $H > \frac{1}{2}$ on a real and separable Hilbert space Y ; and the control function $u(\cdot)$ takes values in $L^2([0, T], U)$, the Hilbert space of admissible control functions for a separable Hilbert space U ; and B is a bounded linear operator from U into X .

For potential applications in telecommunications networks, finance markets, biology and other fields [7, 14], stochastic differential equations driven by fractional Brownian motion have attracted researcher's great interest. Especially, we mention here the recent papers [15, 16, 17, 22]. Moreover, Dung studied the existence and uniqueness of impulsive stochastic Volterra integro-differential equation driven by fBm in [9]. Using the Riemann-Stieltjes integral, Boufoussi et al. [4] proved the existence and uniqueness of a mild solution to a related problem and studied the dependence of the solution on the initial condition in infinite dimensional space. More recently, Li [18] investigated the existence of mild solution to a class of stochastic delay fractional evolution equations driven by fBm. Caraballo et al. [6], and Boufoussi and Hajji [5] have discussed the existence, uniqueness and exponential asymptotic behavior of mild solutions by using the Wiener integral.

To the best of the author's knowledge, an investigation concerning the controllability for fractional neutral stochastic differential equations with infinite delay of the form (1.1) driven by a fractional Brownian motion has not yet been conducted. Thus, we will make the first attempt to study such problem in this paper. Our results are motivated by those in [15, 17] where the controllability of mild solutions to neutral stochastic functional integro-differential equations driven by fractional Brownian motion with finite delays are studied.

The outline of this paper is as follows: In the next section, some necessary notations and concepts are provided. In Section 3, we derive the controllability of fractional neutral stochastic differential systems driven by a fractional Brownian motion. Finally, in Section 4, we conclude with an example to illustrate the applicability of the general theory.

2. PRELIMINARIES

We collect some notions, concepts and lemmas concerning the Wiener integral with respect to an infinite dimensional fractional Brownian, and we recall some basic results which will be used throughout the whole of this paper.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A standard fractional Brownian motion (fBm) $\{\beta^H(t), t \in \mathbb{R}\}$ with Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process with continuous sample paths such that

$$R_H(t, s) = \mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad s, t \in \mathbb{R}. \quad (2.1)$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y, X)$ be the space of bounded linear operator from Y to X . For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $\mathcal{L}(Y, X)$. Let $Q \in \mathcal{L}(Y, Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $\text{tr}Q = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n \geq 0$ ($n = 1, 2, \dots$) are non-negative real numbers and $\{e_n\}$ ($n = 1, 2, \dots$) is a complete orthonormal basis in Y .

We define the infinite dimensional fBm on Y with covariance Q as

$$B^H(t) = B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where β_n^H are real, independent fBm's. This process is Gaussian, it starts from 0, has zero mean and covariance:

$$E\langle B^H(t), x \rangle \langle B^H(s), y \rangle = R(s, t) \langle Q(x), y \rangle \quad \text{for all } x, y \in Y \text{ and } t, s \in [0, T]$$

In order to define Wiener integrals with respect to the Q -fBm, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y, X)$ of all Q -Hilbert-Schmidt operators $\psi : Y \rightarrow X$. We recall that $\psi \in \mathcal{L}(Y, X)$ is called a Q -Hilbert-Schmidt operator, if

$$\|\psi\|_{\mathcal{L}_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \psi e_n\|^2 < \infty,$$

and that the space \mathcal{L}_2^0 equipped with the inner product $\langle \varphi, \psi \rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$ is a separable Hilbert space.

Let $\phi(s); s \in [0, T]$ be a function with values in $\mathcal{L}_2^0(Y, X)$, such that $\sum_{n=1}^{\infty} \|K^* \phi Q^{\frac{1}{2}} e_n\|_{\mathcal{L}_2^0}^2 < \infty$. The Wiener integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n d\beta_n^H(s). \quad (2.2)$$

Now, we end this subsection by stating the following result which is fundamental to prove our result. It can be proved by similar arguments as those used to prove Lemma 2 in [6].

Lemma 2.1. *If $\psi : [0, T] \rightarrow \mathcal{L}_2^0(Y, X)$ satisfies $\int_0^T \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds < \infty$, then the above sum in (2.2) is well defined as a X -valued random variable and we have*

$$\mathbb{E} \left\| \int_0^t \psi(s) dB^H(s) \right\|^2 \leq 2Ht^{2H-1} \int_0^t \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds.$$

It is known that the study of theory of differential equation with infinite delays depends on a choice of the abstract phase space. We assume that the phase space \mathcal{B}_h is a linear space of functions mapping $(-\infty, 0]$ into X , endowed with a norm $\|\cdot\|_{\mathcal{B}_h}$. We shall introduce some basic definitions, notations and lemma which are used in this paper. First, we present the abstract phase space \mathcal{B}_h . Assume that $h : (-\infty, 0] \rightarrow [0, +\infty)$ is a continuous function with $l = \int_{-\infty}^0 h(s) ds < +\infty$.

We define the abstract phase space \mathcal{B}_h by

$$\mathcal{B}_h = \left\{ \psi : (-\infty, 0] \rightarrow X \text{ for any } \tau > 0, (\mathbb{E}\|\psi\|^2)^{\frac{1}{2}} \text{ is bounded and measurable function on } [-\tau, 0] \text{ and } \int_{-\infty}^0 h(t) \sup_{t \leq s \leq 0} (\mathbb{E}\|\psi(s)\|^2)^{\frac{1}{2}} dt < +\infty \right\}.$$

If we equip this space with the norm

$$\|\psi\|_{\mathcal{B}_h} := \int_{-\infty}^0 h(t) \sup_{t \leq s \leq 0} (\mathbb{E}\|\psi(s)\|^2)^{\frac{1}{2}} dt,$$

then it is clear that $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space.

Next, We consider the space \mathcal{B}_T , given by

$$\mathcal{B}_T = \{x : x \in \mathcal{C}((-\infty, T], X), \text{ with } x_0 = \varphi \in \mathcal{B}_h\},$$

where $\mathcal{C}((-\infty, T], X)$ denotes the space of all continuous X -valued stochastic processes $\{x(t), t \in (-\infty, T]\}$. The function $\|\cdot\|_{\mathcal{B}_T}$ to be a semi-norm in \mathcal{B}_T , it is defined by

$$\|x\|_{\mathcal{B}_T} = \|x_0\|_{\mathcal{B}_h} + \sup_{0 \leq t \leq T} (\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}}.$$

The following lemma is a common property of phase spaces.

Lemma 2.2. [19] *Suppose $x \in \mathcal{B}_T$, then for all $t \in [0, T]$, $x_t \in \mathcal{B}_h$ and*

$$l(\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}} \leq \|x_t\|_{\mathcal{B}_h} \leq l \sup_{0 \leq s \leq t} (\mathbb{E}\|x(s)\|^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{B}_h},$$

where $l = \int_{-\infty}^0 h(s) ds < \infty$.

Let us give the following well-known definitions related to fractional order differentiation and integration.

Definition 2.3. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \rightarrow X$ is defined by*

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.4. *The Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ of a function $f : \mathbb{R}^+ \rightarrow X$ is defined by*

$$D_t^\alpha f(t) = \frac{d}{dt} J_t^{1-\alpha} f(t).$$

Definition 2.5. The Caputo fractional derivative of order $\alpha \in (0, 1)$ of $f : \mathbb{R}^+ \rightarrow X$ is defined by

$${}^C D_t^\alpha f(t) = D_t^\alpha (f(t) - f(0)).$$

For more details on fractional calculus, one can see [11].

We suppose $0 \in \rho(A)$, the resolvent set of A , and the semigroup, $(S(t))_{t \geq 0}$, is uniformly bounded. That is, there exists $M \geq 1$ such that $\|S(t)\| \leq M$ for every $t \geq 0$. Then it is possible to define the fractional power $(-A)^\alpha$ for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X , and the expression

$$\|h\|_\alpha = \|(-A)^\alpha h\|$$

defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\|\cdot\|_\alpha$, then the following properties hold (see [21], p. 74).

Lemma 2.6. Suppose that A, X_α , and $(-A)^\alpha$ are as described above.

- (i) For $0 < \alpha \leq 1$, X_α is a Banach space.
- (ii) If $0 < \beta \leq \alpha$, then the injection $X_\alpha \hookrightarrow X_\beta$ is continuous.
- (iii) For every $0 < \alpha \leq 1$, there exists $M_\alpha > 0$ such that

$$\|(-A)^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-\lambda t}, \quad t > 0, \quad \lambda > 0.$$

3. CONTROLLABILITY RESULT

Before starting and proving our main result, we introduce the concepts of a mild solution of the problem (1.1) and the meaning of controllability of fractional neutral stochastic functional differential equation.

Definition 3.1. An X -valued process $\{x(t) : t \in (-\infty, T]\}$ is a mild solution of (1.1) if

- (1) $x(t)$ is continuous on $[0, T]$ almost surely and for each $s \in [0, t)$ and $\alpha \in (0, 1)$ the function $(t-s)^{\alpha-1} A S_\alpha(t-s) g(s, x_s)$ is integrable,
- (2) for arbitrary $t \in [0, T]$, we have

$$\begin{aligned} x(t) &= T_\alpha(t)(\varphi(0) - g(0, \varphi)) + g(t, x_t) \\ &+ \int_0^t (t-s)^{\alpha-1} A S_\alpha(t-s) g(s, x_s) ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, x_s) ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B u(s) ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \sigma(s) dB^H(s), \quad \mathbb{P} - a.s. \end{aligned} \quad (3.1)$$

- (3) $x(t) = \varphi(t)$ on $(-\infty, 0]$ satisfying $\|\varphi\|_{\mathcal{B}_h}^2 < \infty$,

where

$$\begin{aligned} T_\alpha(t)x &= \int_0^\infty \eta_\alpha(\theta) S(t^\alpha \theta) x d\theta, \quad t \geq 0, \quad x \in X. \\ S_\alpha(t)x &= \alpha \int_0^\infty \theta \eta_\alpha(\theta) S(t^\alpha \theta) x d\theta, \quad t \geq 0, \quad x \in X, \end{aligned}$$

where

$$\eta_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \omega_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\omega_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-\alpha n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\alpha\pi), \quad \theta \in]0, \infty[,$$

η_α is a probability density function defined on $(0, \infty)$.

Remark 3.2. (see [27])

$$\int_0^\infty \theta \eta_\alpha(\theta) d\theta = \frac{1}{\Gamma(1 + \alpha)}. \quad (3.2)$$

The following properties of T_α and S_α appeared in [27] are useful.

Lemma 3.3. *Under the previous assumptions on $S(t)$, $t \geq 0$ and A , the operators $T_\alpha(t)$ and $S_\alpha(t)$ have the following properties:*

- (i) For any $x \in X$, $\|T_\alpha(t)x\| \leq M\|x\|$, $\|S_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|$.
- (ii) $\{T_\alpha(t), t \geq 0\}$ and $\{S_\alpha(t), t \geq 0\}$ are strongly continuous.
- (iii) For any $t > 0$, $T_\alpha(t)$ and $S_\alpha(t)$ are also compact operators if $S(t)$ is compact.
- (iv) For any $x \in X$, $\beta \in (0, 1)$ and $\delta \in (0, 1]$, we have

$$AS_\alpha(t)x = A^{1-\beta}S_\alpha A^\beta x, \text{ and } \|A^\delta S_\alpha(t)\| \leq \frac{\alpha M_\delta}{t^{\alpha\delta}} \frac{\Gamma(2-\delta)}{\Gamma(1+\alpha(1-\delta))}, \quad t \in (0, T].$$

Definition 3.4. *The fractional neutral stochastic functional differential equation (1.1) is said to be controllable on the interval $(-\infty, T]$ if for every initial stochastic process φ defined on $(-\infty, 0]$, there exists a stochastic control $u \in L^2([0, T], U)$ such that the mild solution $x(\cdot)$ of (1.1) satisfies $x(T) = x_1$, where x_1 and T are the preassigned terminal state and time, respectively.*

Our main result in this paper is based on the following fixed point theorem.

Theorem 3.5. (Karasnoselskii's fixed point theorem) *Let V be a bounded closed and convex subset of a Banach space X and let Π_1, Π_2 be two operators of V into X satisfying:*

- (1) $\Pi_1(x) + \Pi_2(x) \in V$ whenever $x \in V$,
- (2) Π_1 is a contraction mapping, and
- (3) Π_2 is completely continuous.

Then, there exists a $z \in V$ such that $z = \Pi_1(z) + \Pi_2(z)$.

In order to establish the controllability of (1.1), we impose the following conditions on the data of the problem:

- (H.1) The analytic semigroup, $(S(t))_{t \geq 0}$, generated by A is compact for $t > 0$, and there exists $M \geq 1$ such that

$$\sup_{t \geq 0} \|S(t)\| \leq M, \quad \text{and } c_1 = \|(-A)^{-\beta}\|.$$

- (H.2) The map $f : [0, T] \times \mathcal{B}_h \rightarrow X$ satisfies the following conditions:

- (i) The function $t \mapsto f(t, x)$ is measurable for each $x \in \mathcal{B}_h$, the function $x \mapsto f(t, x)$ is continuous for almost all $t \in [0, T]$,
- (ii) there exists a nonnegative function $p \in L^1([0, T], \mathbb{R}^+)$, and a continuous nondecreasing function $\vartheta : \mathbb{R}^+ \rightarrow (0, +\infty)$ such that for $\delta > \frac{1}{2\alpha-1}$, $(\alpha \in (\frac{1}{2}, 1))$,

$$\int_0^T (\vartheta(s))^\delta ds < \infty, \quad \liminf_{k \rightarrow +\infty} \frac{\vartheta(k)}{k} = \gamma < \infty,$$

and

$$\|f(t, x)\|^2 \leq p(t)\vartheta(\|x\|_{\mathcal{B}_h}^2), \quad \text{for all } x \in \mathcal{B}_h, \text{ almost surely and for a.e. } t \in [0, T].$$

(H.3) The function $g : [0, T] \times \mathcal{B}_h \longrightarrow X$ is continuous. For $\beta \in (0, 1)$, satisfied with $\alpha\beta > \frac{1}{2}$, the function g is X_β -valued and there exists positive constant M_g , such that

$$\|(-A)^\beta g(t, x) - (-A)^\beta g(t, y)\|^2 \leq M_g \|x - y\|_{\mathcal{B}_h}^2, \text{ for all } x \in \mathcal{B}_h, \text{ almost surely and for a.e. } t \in [0, T],$$

$$\|(-A)^\beta g(t, x)\|^2 \leq M_g [\|x\|_{\mathcal{B}_h}^2 + 1], \text{ for all } x \in \mathcal{B}_h, \text{ almost surely and for a.e. } t \in [0, T].$$

(H.4) There exists a constant $p > \frac{1}{2\alpha-1}$ such that the function $\sigma : [0, \infty) \rightarrow \mathcal{L}_2^0(Y, X)$ satisfies

$$\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^{2p} ds < \infty, \quad \forall T > 0.$$

(H.5) The linear operator W from U into X defined by

$$Wu = \int_0^T (T-s)^{\alpha-1} S_\alpha(T-s) Bu(s) ds$$

has an inverse operator W^{-1} that takes values in $L^2([0, T], U) \setminus \ker W$, where

$$\ker W = \{x \in L^2([0, T], U) : Wx = 0\}$$

(see [12]), and there exists finite positive constants M_b, M_w such that $\|B\|^2 \leq M_b$ and $\|W^{-1}\|^2 \leq M_w$.

(H.6) Assume the following inequality holds:

$$\begin{aligned} & 24l^2 \left\{ [c_1^2 + \frac{T^{2\alpha\beta} \alpha^2 M_{1-\beta}^2 \Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)}] M_g + \gamma(1 + \frac{6M^2 M_b M_w T^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}) \frac{M^2 T}{\Gamma^2(\alpha)} \int_0^T (T-s)^{2\alpha-2} p(s) ds \right. \\ & \left. + \frac{6M^2 M_b M_w T^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)} [c_1^2 + \frac{\alpha^2 M_{1-\beta}^2 T^{2\alpha\beta} \Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)}] M_g \right\} < 1. \end{aligned} \quad (3.3)$$

The main result of this chapter is the following.

Theorem 3.6. *Suppose that (H.1) – (H.6) hold. Then, the system (1.1) is controllable on $(-\infty, T]$.*

Proof. Transform the problem(1.1) into a fixed-point problem. To do this, using the hypothesis (H.5) for an arbitrary function $x(\cdot)$, define the control by

$$\begin{aligned} u(t) &= W^{-1} \{ x_1 - T_\alpha(T) [\varphi(0) - g(0, x_0)] - g(T, x_T) \} \\ &- \int_0^T (T-s)^{\alpha-1} A S_\alpha(T-s) g(s, x_s) ds - \int_0^T (T-s)^{\alpha-1} S_\alpha(T-s) f(s, x_s) ds \\ &- \int_0^T (T-s)^{\alpha-1} S_\alpha(T-s) \sigma(s) dB^H(s) \} (t), \quad t \in [0, T]. \end{aligned} \quad (3.4)$$

To formulate the controllability problem in the form suitable for application of the fixed point theorem, put the control $u(\cdot)$ into the stochastic control system (3.1) and obtain a non

linear operator Π on \mathcal{B}_T given by

$$\Pi(x)(t) = \begin{cases} \varphi(t), & \text{if } t \in (-\infty, 0], \\ T_\alpha(t)(\varphi(0) - g(0, \varphi)) + g(t, x_t) + \int_0^t (t-s)^{\alpha-1} AS_\alpha(t-s)g(s, x_s)ds \\ + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)f(s, x_s)ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)Bu(s)ds \\ + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\sigma(s)dB^H(s), & \text{if } t \in [0, T]. \end{cases}$$

Then it is clear that to prove the existence of mild solutions to equation (1.1) is equivalent to find a fixed point for the operator Π . Clearly, $\Pi x(T) = x_1$, which means that the control u steers the system from the initial state φ to x_1 in time T , provided we can obtain a fixed point of the operator Π which implies that the system is controllable.

Let $y : (-\infty, T] \rightarrow X$ be the function defined by

$$y(t) = \begin{cases} \varphi(t), & \text{if } t \in (-\infty, 0], \\ S(t)\varphi(0), & \text{if } t \in [0, T], \end{cases}$$

then, $y_0 = \varphi$. For each function $z \in \mathcal{B}_T$, set

$$x(t) = z(t) + y(t).$$

It is obvious that x satisfies the stochastic control system (3.1) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) = & g(t, z_t + y_t) - T_\alpha(t)g(0, \varphi) + \int_0^t (t-s)^{\alpha-1} AS_\alpha(t-s)g(s, z_s + y_s)ds \\ & + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)f(s, z_s + y_s)ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)Bu_{z+y}(s)ds \\ & + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\sigma(s)dB^H(s), \end{aligned} \quad (3.5)$$

where $u_{z+y}(t)$ is obtained from (3.4) by replacing $x_t = z_t + y_t$.

Set

$$\mathcal{B}_T^0 = \{z \in \mathcal{B}_T : z_0 = 0\};$$

for any $z \in \mathcal{B}_T^0$, we have

$$\|z\|_{\mathcal{B}_T^0} = \|z_0\|_{\mathcal{B}_h} + \sup_{t \in [0, T]} (\mathbb{E}\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t \in [0, T]} (\mathbb{E}\|z(t)\|^2)^{\frac{1}{2}}.$$

Then, $(\mathcal{B}_T^0, \|\cdot\|_{\mathcal{B}_T^0})$ is a Banach space. Define the operator $\widehat{\Pi} : \mathcal{B}_T^0 \rightarrow \mathcal{B}_T^0$ by

$$(\widehat{\Pi}z)(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0], \\ g(t, z_t + y_t) - T_\alpha(t)g(0, \varphi) + \int_0^t (t-s)^{\alpha-1} AS_\alpha(t-s)g(s, z_s + y_s)ds \\ + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)f(s, z_s + y_s)ds \\ + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)Bu_{z+y}(s)ds \\ + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\sigma(s)dB^H(s), & \text{if } t \in [0, T]. \end{cases} \quad (3.6)$$

Set

$$\mathcal{B}_k = \{z \in \mathcal{B}_T^0 : \|z\|_{\mathcal{B}_T^0}^2 \leq k\}, \quad \text{for some } k \geq 0,$$

then $\mathcal{B}_k \subseteq \mathcal{B}_T^0$ is a bounded closed convex set, and for $z \in \mathcal{B}_k$, we have

$$\begin{aligned} \|z_t + y_t\|_{\mathcal{B}_h}^2 &\leq 2(\|z_t\|_{\mathcal{B}_h}^2 + \|y_t\|_{\mathcal{B}_h}^2) \\ &\leq 4(l^2 \sup_{0 \leq s \leq t} \mathbb{E}\|z(s)\|^2 + \|z_0\|_{\mathcal{B}_h}^2 \\ &\quad + l^2 \sup_{0 \leq s \leq t} \mathbb{E}\|y(s)\|^2 + \|y_0\|_{\mathcal{B}_h}^2) \\ &\leq 4l^2(k + M^2 \mathbb{E}\|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h}^2 \\ &:= q'. \end{aligned} \quad (3.7)$$

It is clear that the operator Π has a fixed point if and only if $\widehat{\Pi}$ has one, so it turns to prove that $\widehat{\Pi}$ has a fixed point. To this end, we decompose $\widehat{\Pi}$ as $\widehat{\Pi} = \Pi_1 + \Pi_2$, where Π_1 and Π_2 are defined on \mathcal{B}_T^0 , respectively by

$$(\Pi_1 z)(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0], \\ g(t, z_t + y_t) - T_\alpha(t)g(0, \varphi) + \int_0^t (t-s)^{\alpha-1} AS_\alpha(t-s)g(s, z_s + y_s)ds \\ + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\sigma(s)dB^H(s), & \text{if } t \in [0, T], \end{cases} \quad (3.8)$$

and

$$(\Pi_2 z)(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0], \\ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)f(s, z_s + y_s)ds \\ + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)Bu_{z+y}(s)ds, & \text{if } t \in [0, T]. \end{cases} \quad (3.9)$$

For the sake of convenience, the proof will be given in several steps.

Step 1. We claim that there exists a positive number k , such that $\Pi_1(x) + \Pi_2(x) \in \mathcal{B}_k$ whenever $x \in \mathcal{B}_k$. If it is not true, then for each positive number k , there is a function $z^k(\cdot) \in \mathcal{B}_k$, but $\Pi_1(z^k) + \Pi_2(z^k) \notin \mathcal{B}_k$, that is $\mathbb{E}\|\Pi_1(z^k)(t) + \Pi_2(z^k)(t)\|^2 > k$ for some

$t \in [0, T]$. However, on the other hand, we have

$$\begin{aligned}
k < \mathbb{E} \|\Pi_1(z^k)(t) + \Pi_2(z^k)(t)\|^2 &\leq 6\{\mathbb{E}\|T_\alpha(t)g(0, \varphi)\|^2 + \mathbb{E}\|g(t, z_t^k + y_t)\|^2 \\
&\quad + \mathbb{E}\|\int_0^t (t-s)^{\alpha-1} AS_\alpha(t-s)g(s, z_s^k + y_s)ds\|^2 \\
&\quad + \mathbb{E}\|\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)f(s, z_s^k + y_s)ds\|^2 \\
&\quad + \mathbb{E}\|\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)Bu_{z^k+y}(s)ds\|^2 \\
&\quad + \mathbb{E}\|\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\sigma(s)dB^H(s)\|^2\} \\
&\leq 6\sum_{i=1}^6 I_i.
\end{aligned} \tag{3.10}$$

By $(\mathcal{H}.3)$, (i) of Lemma 3.3, we have

$$\begin{aligned}
I_1 &\leq \mathbb{E}\|T_\alpha(t)g(0, \varphi)\|^2 \\
&\leq M^2\|(-A)^{-\beta}\|^2\|(-A)^\beta g(0, \varphi)\|^2 \\
&\leq M^2 c_1^2 M_g [\|\varphi\|_{\mathcal{B}_h}^2 + 1].
\end{aligned} \tag{3.11}$$

By $(\mathcal{H}.3)$, (3.7), we have

$$\begin{aligned}
I_2 &\leq \|(-A)^{-\beta}\|^2 \mathbb{E}\|(-A)^\beta g(t, z_t^k + y_t)\|^2 \\
&\leq c_1^2 M_g [\|z_t^k + y_t\|_{\mathcal{B}_h}^2 + 1] \\
&\leq c_1^2 M_g [4l^2(k + M^2 \mathbb{E}\|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h}^2 + 1].
\end{aligned} \tag{3.12}$$

By (iv) of Lemma 3.3, $(\mathcal{H}.3)$, Hölder inequality, we have

$$\begin{aligned}
I_3 &\leq \mathbb{E}\|\int_0^t (t-s)^{\alpha-1} AS_\alpha(t-s)g(s, z_s^k + y_s)ds\|^2 \\
&\leq \mathbb{E}\|(\int_0^t (t-s)^{\alpha-1} (-A)^{1-\beta} S_\alpha(t-s) (-A)^\beta g(s, z_s^k + y_s)ds\|^2 \\
&\leq \mathbb{E}(\int_0^t (t-s)^{\alpha-1} \|(-A)^{1-\beta} S_\alpha(t-s) (-A)^\beta g(s, z_s^k + y_s)\| ds)^2 \\
&\leq \frac{\alpha^2 M_{1-\beta}^2 \Gamma^2(\beta+1)}{\Gamma^2(\alpha\beta+1)} \mathbb{E}(\int_0^t (t-s)^{\alpha-1} \|(t-s)^{\alpha\beta-\alpha} (-A)^\beta g(s, z_s^k + y_s)\| ds)^2 \\
&\leq \frac{\alpha^2 M_{1-\beta}^2 \Gamma^2(\beta+1)}{\Gamma^2(\alpha\beta+1)} \int_0^t (t-s)^{2\alpha\beta-2} ds \int_0^t \mathbb{E}\|(-A)^\beta g(s, z_s^k + y_s)\|^2 ds \\
&\leq \frac{T^{2\alpha\beta-1} \alpha^2 M_{1-\beta}^2 \Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)} \int_0^t M_g (4l^2(k + M^2 \mathbb{E}\|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h}^2 + 1) ds \\
&\leq \frac{T^{2\alpha\beta} \alpha^2 M_{1-\beta}^2 \Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)} M_g [4l^2(k + M^2 \mathbb{E}\|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h}^2 + 1].
\end{aligned} \tag{3.13}$$

From $(\mathcal{H}.2)$, Hölder inequality, we have

$$\begin{aligned}
I_4 &\leq \mathbb{E} \left\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, z_s^k + y_s) ds \right\|^2 \\
&\leq \frac{M^2 T}{\Gamma^2(\alpha)} \mathbb{E} \int_0^t \|(t-s)^{\alpha-1} f(s, z_s^k + y_s)\|^2 ds \\
&\leq \frac{M^2 T}{\Gamma^2(\alpha)} \int_0^T (T-s)^{2\alpha-2} \mathbb{E} \|f(s, z_s^k + y_s)\|^2 ds \\
&\leq \frac{M^2 T}{\Gamma^2(\alpha)} \int_0^T (T-s)^{2\alpha-2} p(s) \vartheta(\|z_s^k + y_s\|_{\mathcal{B}_h}^2) ds \\
&\leq \frac{M^2 T}{\Gamma^2(\alpha)} \vartheta(4l^2(k + M^2 \mathbb{E} \|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h}^2) \int_0^T (T-s)^{2\alpha-2} p(s) ds
\end{aligned} \tag{3.14}$$

From (ii) of $(\mathcal{H}.2)$, Hölder inequality, it follows that for $\delta > \frac{1}{2\alpha-1}$,

$$\begin{aligned}
\int_0^T (T-s)^{2\alpha-2} p(s) ds &\leq \left(\int_0^T (T-s)^{\frac{(2\alpha-2)\delta}{\delta-1}} ds \right)^{\frac{\delta-1}{\delta}} \left(\int_0^T (p(s))^\delta ds \right)^{\frac{1}{\delta}} \\
&\leq T^{\frac{(2\alpha-1)\delta-1}{\delta}} \left(\int_0^T (p(s))^\delta ds \right)^{\frac{1}{\delta}} \\
&< \infty.
\end{aligned}$$

From our assumptions, (iv) of Lemma 3.3, using the fact that $(\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2$ for any positive real numbers $a_i, i = 1, 2, \dots, n$, we have

$$\begin{aligned}
\mathbb{E} \|u_{z+y}\|^2 &\leq 6M_w \{ \|x_1\|^2 + M^2 \mathbb{E} \|\varphi(0)\|^2 + M^2 c_1^2 M_g [\|y\|_{\mathcal{B}_h}^2 + 1] \\
&\quad + [c_1^2 + \frac{\alpha^2 M_{1-\beta}^2 T^{2\alpha\beta} \Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)}] M_g [4l^2(k + M^2 \mathbb{E} \|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h}^2 + 1] \\
&\quad + \frac{M^2}{\Gamma^2(\alpha)} \vartheta(4l^2(k + M^2 \mathbb{E} \|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h}^2) \int_0^T (T-s)^{2\alpha-2} p(s) ds \\
&\quad + 2 \frac{M^2}{\Gamma^2(\alpha)} T^{2H-1} \int_0^T (T-s)^{(2\alpha-2)} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds \} := \mathcal{G}.
\end{aligned} \tag{3.15}$$

For $p > \frac{1}{2\alpha-1}$, we have

$$\begin{aligned}
\int_0^T (T-s)^{(2\alpha-2)} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds &\leq \left(\int_0^T (T-s)^{\frac{(2\alpha-2)p}{p-1}} ds \right)^{\frac{p-1}{p}} \left(\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^{2p} ds \right)^{\frac{1}{p}} \\
&\leq T^{\frac{(2\alpha-1)p-1}{p}} \left(\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^{2p} ds \right)^{\frac{1}{p}} \\
&< \infty.
\end{aligned} \tag{3.16}$$

By (3.15), (i) of Lemma 3.3, Hölder inequality, we have

$$\begin{aligned}
I_5 &\leq \mathbb{E} \left\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B u_{z^k+y}(s) ds \right\|^2 \\
&\leq \frac{M^2 M_b}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2\alpha-2} ds \int_0^t \mathbb{E} \|u_{z^k+y}(s)\|^2 ds \\
&\leq \frac{6M^2 M_b M_w T^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)} \{ \|x_1\|^2 + M^2 \mathbb{E} \|\varphi(0)\|^2 + M^2 c_1^2 M_g [\|y\|_{\mathcal{B}_h}^2 + 1] \\
&\quad + [c_1^2 + \frac{\alpha^2 M_{1-\beta}^2 T^{2\alpha\beta} \Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)}] M_g [4l^2(k + M^2 \mathbb{E} \|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h}^2 + 1] \\
&\quad + \frac{M^2}{\Gamma^2(\alpha)} \vartheta(4l^2(k + M^2 \mathbb{E} \|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h}^2) \int_0^T (T-s)^{2\alpha-2} p(s) ds \\
&\quad + 2 \frac{M^2}{\Gamma^2(\alpha)} T^{2H-1} \int_0^T (T-s)^{(2\alpha-2)} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds \}.
\end{aligned} \tag{3.17}$$

By Lemma 2.1, Lemma 3.3, (3.16), for $p > \frac{1}{2\alpha-1}$, we have

$$\begin{aligned}
I_6 &\leq \mathbb{E} \left\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \sigma(s) dB^H(s) \right\|^2 \\
&\leq \frac{2M^2 T^{2H-1}}{\Gamma^2(\alpha)} \int_0^T (T-s)^{(2\alpha-2)} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds \\
&\leq \frac{2M^2 T^{2H-1}}{\Gamma^2(\alpha)} T^{\frac{(2\alpha-1)p-1}{p}} \left(\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^{2p} ds \right)^{\frac{1}{p}}.
\end{aligned} \tag{3.18}$$

By (3.10), (3.11), (3.12), (3.13), (3.14), (3.17), (3.18), we have

$$\begin{aligned}
k &< \mathbb{E} \|\Pi_1(z^k)(t) + \Pi_2(z^k)(t)\|^2 \leq \overline{K} + 24l^2 k c_1^2 M_g + 24l^2 k \frac{T^{2\alpha\beta} \alpha^2 M_{1-\beta}^2 \Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)} M_g \\
&\quad + 6(1 + \frac{6M^2 M_b M_w T^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}) \frac{M^2 T}{\Gamma^2(\alpha)} \vartheta(4l^2(k + M^2 \mathbb{E} \|\varphi(0)\|^2) \\
&\quad + 4\|y\|_{\mathcal{B}_h}^2) \int_0^T (T-s)^{2\alpha-2} p(s) ds \\
&\quad + \frac{144M^2 M_b M_w T^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)} [c_1^2 + \frac{\alpha^2 M_{1-\beta}^2 T^{2\alpha\beta} \Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)}] M_g l^2 k,
\end{aligned}$$

where

$$\begin{aligned}
\overline{K} &= 6M^2 c_1^2 (M_g \|\varphi\|_{\mathcal{B}_h}^2 + 6c_1^2 M_g [4l^2 M^2 \mathbb{E} \|\varphi(0)\|^2 + 4\|y\|_{\mathcal{B}_h}^2 + 1] \\
&\quad + 6 \frac{T^{2\alpha\beta} \alpha^2 M_{1-\beta}^2 \Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)} M_g [4l^2 M^2 \mathbb{E} \|\varphi(0)\|^2 + 4\|y\|_{\mathcal{B}_h}^2 + 1] \\
&\quad + \frac{36M^2 M_b M_w T^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)} \{ \|x_1\|^2 + M^2 \mathbb{E} \|\varphi(0)\|^2 + M^2 c_1^2 M_g [\|y\|_{\mathcal{B}_h}^2 + 1] \\
&\quad + \frac{6M^2 M_b M_w T^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)} [c_1^2 + \frac{\alpha^2 M_{1-\beta}^2 T^{2\alpha\beta} \Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)}] M_g [4l^2 M^2 \mathbb{E} \|\varphi(0)\|^2 + 4\|y\|_{\mathcal{B}_h}^2 + 1] \} \\
&\quad + 6(1 + \frac{6M^2 M_b M_w T^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}) \frac{2M^2 T^{2H-1}}{\Gamma^2(\alpha)} T^{\frac{(2\alpha-1)p-1}{p}} \left(\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^{2p} ds \right)^{\frac{1}{p}}.
\end{aligned}$$

Noting that \overline{K} is independent of k . Dividing both sides by k and taking the lower limit as $k \rightarrow \infty$, we obtain

$$q' = 4l^2(k + M\mathbb{E}\|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h} \rightarrow \infty \text{ as } k \rightarrow \infty,$$

$$\liminf_{k \rightarrow \infty} \frac{\vartheta(q')}{k} = \liminf_{k \rightarrow \infty} \frac{\vartheta(q')}{q'} \cdot \frac{q'}{k} = 4l^2\gamma.$$

Thus, we have

$$\begin{aligned} 1 &\leq 24l^2c_1^2M_g + 24l^2 \frac{T^{2\alpha\beta}\alpha^2M_{1-\beta}^2\Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)}M_g \\ &\quad + 24l^2\gamma(1 + \frac{6M^2M_bM_wT^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}) \frac{M^2T}{\Gamma^2(\alpha)} \int_0^T (T-s)^{2\alpha-2}p(s)ds \\ &\quad + \frac{144M^2M_bM_wT^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)} [c_1^2 + \frac{\alpha^2M_{1-\beta}^2T^{2\alpha\beta}\Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)}] M_g l^2. \end{aligned}$$

This contradicts (3.3). Hence for some positive k ,

$$(\Pi_1 + \Pi_2)(\mathcal{B}_k) \subseteq \mathcal{B}_k.$$

Step 2. Π_1 is a contraction.

Let $t \in [0, T]$ and $z^1, z^2 \in \mathcal{B}_T^0$

$$\begin{aligned} \mathbb{E}\|(\Pi_1 z^1)(t) - (\Pi_1 z^2)(t)\|^2 &\leq 2\mathbb{E}\|g(t, z_t^1 + y_t) - g(t, z_t^2 + y_t)\|^2 \\ &\quad + 2\mathbb{E}\|\int_0^t (t-s)^{\alpha-1} AS_\alpha(t-s)(g(s, z_s^1 + y_s) - g(s, z_s^2 + y_s))ds\|^2 \\ &\leq 2M_g \|(-A)^{-\beta}\|^2 \|z_s^1 - z_s^2\|_{\mathcal{B}_h}^2 \\ &\quad + 2\int_0^t (t-s)^{\alpha-1} (-A)^{1-\beta} S_\alpha(t-s) (-A)^\beta (g(s, z_s^1 + y_s) - g(s, z_s^2 + y_s))ds\|^2 \\ &\leq 2M_g \|(-A)^{-\beta}\|^2 \|z_s^1 - z_s^2\|_{\mathcal{B}_h}^2 \\ &\quad + \frac{2\alpha^2 M_{1-\beta}^2 \Gamma^2(\beta+1)}{\Gamma^2(\alpha\beta+1)} \int_0^t (t-s)^{2\alpha\beta-2} ds \int_0^t M_g \|z_s^1 - z_s^2\|_{\mathcal{B}_h}^2 ds \\ &\leq 2M_g \left\{ \|(-A)^{-\beta}\|^2 + \frac{2\alpha^2 M_{1-\beta}^2 \Gamma^2(\beta+1)}{\Gamma^2(\alpha\beta+1)} \frac{T^{2\alpha\beta}}{2\alpha\beta-1} \right\} (2l^2 \sup_{0 \leq s \leq T} \|z_s^1 - z_s^2\|_{\mathcal{B}_h}^2) \\ &\leq \nu \sup_{0 \leq s \leq T} \mathbb{E}\|z^1(s) - z^2(s)\|^2 \quad (\text{since } z_0^1 = z_0^2 = 0) \end{aligned}$$

Taking supremum over t ,

$$\|(\Pi_1 z^1)(t) - (\Pi_1 z^2)(t)\|_{\mathcal{B}_T^0} \leq \nu \|z^1 - z^2\|_{\mathcal{B}_T^0},$$

where

$$\nu = 4M_g l^2 \left\{ c_1^2 + \frac{2\alpha^2 M_{1-\beta}^2 \Gamma^2(\beta+1)}{\Gamma^2(\alpha\beta+1)} \frac{T^{2\alpha\beta}}{2\alpha\beta-1} \right\}.$$

By (H.6), we have $\nu < 1$. Thus Π_1 is a contraction on \mathcal{B}_T^0 .

Step 3. Π_2 is completely continuous \mathcal{B}_T^0 .

Claim 1. Π_2 is continuous on \mathcal{B}_T^0 .

Let z^n be a sequence such that $z^n \rightarrow z$ in \mathcal{B}_T^0 . Then, for $t \in [0, T]$, and thanks to hypothesis (H.2) – (H.3), for each $t \in [0, T]$, we have

$$f(t, z_t^n + y_t) \rightarrow f(t, z_t + y_t),$$

$$g(t, z_t^n + y_t) \rightarrow g(t, z_t + y_t).$$

By the dominated convergence theorem, we obtain continuity of Π_2

$$\begin{aligned}
\mathbb{E}\|\Pi_2 z^n(t) - (\Pi_2 z)(t)\|^2 &\leq 2\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B[u_{z^n+y} - u_{z+y}] ds\right\|^2 \\
&\quad + 2\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) [f(s, z_s^n + y_s) - f(s, z_s + y_s)] ds\right\|^2 \\
&\leq \frac{2M^2 M_b}{\Gamma^2(\alpha+1)} \frac{T^{2\alpha-1}}{2\alpha-1} \int_0^T \mathbb{E}\|u_{z^n+y}(s) - u_{z+y}(s)\|^2 ds \\
&\quad + \frac{2M^2}{\Gamma^2(\alpha+1)} \frac{T^{2\alpha-1}}{2\alpha-1} \int_0^T \mathbb{E}\|f(s, z_s^n + y_s) - f(s, z_s + y_s)\|^2 ds \\
&\longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Thus, Π_2 is continuous.

Claim 2. Π_2 maps \mathcal{B}_k into equicontinuous family. Let $z \in \mathcal{B}_k$ and $|h|$ be sufficiently small, we have

$$\begin{aligned}
\mathbb{E}\|(\Pi_2 z)(t+h) - (\Pi_2 z)(t)\|^2 &\leq \mathbb{E}\left\|\int_0^{t+h} (t+h-s)^{\alpha-1} S_\alpha(t+h-s) B u_{z+y}(s) ds\right. \\
&\quad + \int_0^{t+h} (t+h-s)^{\alpha-1} S_\alpha(t+h-s) f(s, z_s + y_s) ds \\
&\quad - \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B u_{z+y}(s) ds \\
&\quad \left. - \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, z_s + y_s) ds\right\|^2 \\
&\leq 6\mathbb{E}\left\|\int_0^t ((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}) S_\alpha(t+h-s) B u_{z+y}(s) ds\right\|^2 \\
&\quad + 6\mathbb{E}\left\|\int_t^{t+h} (t+h-s)^{\alpha-1} S_\alpha(t+h-s) B u_{z+y}(s) ds\right\|^2 \\
&\quad + 6\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1} (S_\alpha(t+h-s) - S_\alpha(t-s)) B u_{z+y}(s) ds\right\|^2 \\
&\quad + 6\mathbb{E}\left\|\int_0^t ((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}) S_\alpha(t+h-s) f(s, z_s + y_s) ds\right\|^2 \\
&\quad + 6\mathbb{E}\left\|\int_t^{t+h} (t+h-s)^{\alpha-1} S_\alpha(t+h-s) f(s, z_s + y_s) ds\right\|^2 \\
&\quad + 6\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1} (S_\alpha(t+h-s) - S_\alpha(t-s)) f(s, z_s + y_s) ds\right\|^2.
\end{aligned}$$

From (iii) of Lemma 3.3, we have $S_\alpha(t)$ is compact for any $t > 0$. Let $0 < \varepsilon < t < T$, and $\delta > 0$ such that $\|S_\alpha(\tau_1) - S_\alpha(\tau_2)\| \leq \varepsilon$ for every $\tau_1, \tau_2 \in [0, T]$ with $|\tau_1 - \tau_2| \leq \delta$.

From (3.15), (i) of Lemma 3.3, Hölder inequality, it follows that

$$\begin{aligned}
& \mathbb{E} \|(\Pi_2 z)(t+h) - (\Pi_2 z)(t)\|^2 \\
& \leq \frac{6M^2 M_b \mathcal{G} T}{\Gamma^2(\alpha)} \int_0^t ((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1})^2 ds \\
& \quad + \frac{6M^2 M_b \mathcal{G} h}{\Gamma^2(\alpha)} \int_t^{t+h} (t+h-s)^{2\alpha-2} ds \\
& \quad + \frac{6M^2 T^{2\alpha} \mathcal{G}}{2\alpha-1} \epsilon \\
& \quad + \frac{6M^2 T \vartheta(q')}{\Gamma^2(\alpha)} \int_0^t ((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1})^2 p(s) ds \\
& \quad + \frac{6M^2 T \vartheta(q')}{\Gamma^2(\alpha)} \int_t^{t+h} (t+h-s)^{2(\alpha-1)} p(s) ds \\
& \quad + \frac{6M^2 T}{2\alpha-1} \epsilon \int_0^t (t-s)^{2(\alpha-1)} p(s) ds.
\end{aligned} \tag{3.19}$$

From (ii) of $(\mathcal{H}.2)$, Hölder inequality, it follows that for $\delta > \frac{1}{2\alpha-1}$,

$$\begin{aligned}
\int_0^t (t-s)^{2\alpha-2} p(s) ds & \leq \left(\int_0^t (t-s)^{\frac{(2\alpha-2)\delta}{\delta-1}} ds \right)^{\frac{\delta-1}{\delta}} \left(\int_0^T (p(s))^\delta ds \right)^{\frac{1}{\delta}} \\
& \leq T^{\frac{(2\alpha-1)\delta-1}{\delta}} \left(\int_0^T (p(s))^\delta ds \right)^{\frac{1}{\delta}} \\
& < \infty.
\end{aligned}$$

Similarly, we have

$$\int_0^t (t+h-s)^{2(\alpha-1)} p(s) ds < \infty.$$

By the dominated convergence theorem, we have

$$\int_0^t ((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1})^2 p(s) ds \longrightarrow 0, \text{ as } h \longrightarrow 0.$$

Therefore, for sufficiently small positive number ϵ , we have from (3.19) that

$$\mathbb{E} \|(\Pi_2 z)(t+h) - (\Pi_2 z)(t)\|^2 \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

Thus, Π_2 maps \mathcal{B}_k into an equicontinuous family of functions.

Claim 3. $(\Pi_2 \mathcal{B}_k)(t)$ is precompact set in X .

Let $0 < t \leq T$ be fixed, and ϵ be a number satisfying $0 < \epsilon < t$. For $\delta > 0$ and $z \in \mathcal{B}_k$, we define

$$\begin{aligned}
(\Pi_{2,\epsilon}^\delta z)(t) &= \alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha \theta) f(s, z_s + y_s) d\theta ds \\
&+ \alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha \theta) Bu_{z+y}(s) d\theta ds \\
&= S(\epsilon^\alpha \delta) \alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha \theta - \epsilon^\alpha \delta) f(s, z_s + y_s) d\theta ds \\
&+ S(\epsilon^\alpha \delta) \alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha \theta - \epsilon^\alpha \delta) Bu_{z+y}(s) d\theta ds
\end{aligned}$$

From the compactness of $S(t)$ ($t > 0$), we obtain that the set $V_\epsilon^\delta(t) = \{(\Pi_{2,\epsilon}^\delta z)(t) : z \in \mathcal{B}_k\}$ is relative compact in X for every ϵ , $0 < \epsilon < t$ and $\delta > 0$. Moreover, for every $z \in \mathcal{B}_k$, we have

$$\begin{aligned}
\mathbb{E} \|\Pi_2 z(t) - \Pi_{2,\epsilon}^\delta z(t)\|^2 &\leq 4\alpha^2 \mathbb{E} \left\| \int_0^t \int_0^\delta \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha \theta) f(s, z_s + y_s) d\theta ds \right\|^2 \\
&+ 4\alpha^2 \mathbb{E} \left\| \int_{t-\epsilon}^t \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha \theta) f(s, z_s + y_s) d\theta ds \right\|^2 \\
&+ 4\alpha^2 \mathbb{E} \left\| \int_0^t \int_0^\delta \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha \theta) Bu_{z+y}(s) d\theta ds \right\|^2 \\
&+ 4\alpha^2 \mathbb{E} \left\| \int_{t-\epsilon}^t \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha \theta) Bu_{z+y}(s) d\theta ds \right\|^2 \\
&= 4 \sum_{i=1}^4 J_i.
\end{aligned} \tag{3.20}$$

A similar argument as before, we can show that

$$\begin{aligned}
J_1 &\leq \alpha^2 M^2 T \mathbb{E} \left\| \int_0^t \int_0^\delta \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) f(s, z_s + y_s) d\theta \right\|^2 ds \\
&\leq \alpha^2 M^2 T \left\| \int_0^\delta \theta \eta_\alpha(\theta) d\theta \right\|^2 \int_0^t (t-s)^{2\alpha-2} \mathbb{E} \|f(s, z_s + y_s)\|^2 ds \\
&\leq \alpha^2 M^2 T \vartheta(q') \left\| \int_0^\delta \theta \eta_\alpha(\theta) d\theta \right\|^2 \int_0^t (t-s)^{2\alpha-2} p(s) ds.
\end{aligned} \tag{3.21}$$

For J_2 , by (3.2), we have

$$\begin{aligned}
J_2 &\leq \alpha^2 M^2 T \vartheta(q') \left\| \int_0^\infty \theta \eta_\alpha(\theta) d\theta \right\|^2 \int_{t-\epsilon}^t (t-s)^{2\alpha-2} p(s) ds \\
&\leq \frac{\alpha^2 M^2 T \vartheta(q')}{\Gamma^2(1+\alpha)} \int_{t-\epsilon}^t (t-s)^{2\alpha-2} p(s) ds \\
&\leq \frac{\alpha^2 M^2 T \vartheta(q')}{\Gamma^2(1+\alpha)} \left(\int_{t-\epsilon}^t (t-s)^{\frac{(2\alpha-2)\delta}{\delta-1}} ds \right)^{\frac{\delta-1}{\delta}} \left(\int_{t-\epsilon}^t (p(s))^\delta ds \right)^{\frac{1}{\delta}} \\
&\leq \frac{\alpha^2 M^2 T \vartheta(q')}{\Gamma^2(1+\alpha)} \epsilon^{\frac{(2\alpha-1)\delta-1}{\delta}} \left(\int_{t-\epsilon}^t (p(s))^\delta ds \right)^{\frac{1}{\delta}},
\end{aligned} \tag{3.22}$$

where $\delta > \frac{1}{2\alpha-1}$.

For J_3 , by Hölder inequality, we have

$$\begin{aligned} J_3 &\leq \alpha^2 \mathbb{E} \left(\int_0^t \int_0^\delta \|\theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha \theta) B u_{z+y}(s)\| d\theta ds \right)^2 \\ &\leq \alpha^2 M^2 M_b T \int_0^t (t-s)^{2\alpha-2} \mathbb{E} \|u_{z+y}(s)\|^2 ds \int_0^\delta \theta \eta_\alpha(\theta) d\theta \|^2. \end{aligned} \quad (3.23)$$

For J_4 , by (3.2), we have

$$\begin{aligned} J_4 &\leq \alpha^2 M^2 \mathbb{E} \int_{t-\epsilon}^t \|(t-s)^{\alpha-1} B u_{z+y}(s)\|^2 ds \int_{t-\epsilon}^t \|\int_0^\infty \theta \eta_\alpha(\theta) d\theta\|^2 ds \\ &\leq \frac{\epsilon \alpha^2 M^2 M_b}{\Gamma^2(\alpha+1)} \int_{t-\epsilon}^t (t-s)^{2\alpha-2} \mathbb{E} \|u_{z+y}(s)\|^2 ds \end{aligned} \quad (3.24)$$

Put (3.21), (3.22), (3.23), (3.24) into (3.20) to obtain

$$\mathbb{E} \|\Pi_2 z(t) - \Pi_{2,\epsilon}^\delta z(t)\|^2 \longrightarrow 0, \quad \text{as } \epsilon \longrightarrow 0^+, \delta \longrightarrow 0^+.$$

Therefore, there are precompact sets arbitrarily close to the set $V(t) = \{(\Pi_2 z)(t) : z \in B_k\}$, hence the set $V(t)$ is also precompact in X .

Thus, by Arzela-Ascoli theorem Π_2 is a compact operator. These arguments enable us to conclude that Π_2 is completely continuous, and by the fixed point theorem of Karasnoselskii there exists a fixed point $z(\cdot)$ for $\widehat{\Pi}$ on B_k . If we define $x(t) = z(t) + y(t)$, $-\infty < t \leq T$, it is easy to see that $x(\cdot)$ is a mild solution of (1.1) satisfying $x_0 = \varphi$, $x(T) = x_1$. Then the proof is complete. \square

4. EXAMPLE

To illustrate the previous result, we consider the following fractional neutral stochastic partial differential equation with infinite delays, driven by a fractional Brownian motion of the form

$$\begin{cases} dJ_t^{1-\alpha} [v(t, \xi) - g(t, v(t-r, \xi)) - \varphi(0, \xi) + g(0, v(-r, \xi))] = [\frac{\partial^2}{\partial^2 \xi} v(t, \xi) + c(\xi)u(t) \\ + f(t, t-r, \xi)] dt + \sigma(t) \frac{dB^H(t)}{dt}, \quad 0 \leq t \leq T, r > 0, 0 \leq \xi \leq 1 \\ v(t, 0) = v(t, 1) = 0, \quad 0 \leq t \leq T, \\ v(s, \xi) = \varphi(s, \xi), \quad ; -\infty < s \leq 0 \quad 0 \leq \xi \leq 1, \end{cases} \quad (4.1)$$

where $B^H(t)$ is cylindrical fractional Brownian motion, $\varphi : (-\infty, 0] \times [0, 1] \longrightarrow \mathbb{R}$ is a given measurable and satisfies $\|\varphi\|_{B_h}^2 < \infty$.

We rewrite (4.1) into abstract form of (1.1). We take $X = Y = U = L^2([0, 1])$. Define the operator $A : D(A) \subset X \longrightarrow X$ given by $A = \frac{\partial^2}{\partial^2 \xi}$ with

$$D(A) = \{y \in X : y' \text{ is absolutely continuous, } y'' \in X, \quad y(0) = y(1) = 0\},$$

then we get

$$Ax = \sum_{n=1}^{\infty} n^2 \langle x, e_n \rangle_X e_n, \quad x \in D(A),$$

where $e_n := \sqrt{\frac{2}{\pi}} \sin nx$, $n = 1, 2, \dots$ is an orthogonal set of eigenvector of $-A$.

The bounded linear operator $(-A)^{\frac{2}{3}}$ is given by

$$(-A)^{\frac{2}{3}}x = \sum_{n=1}^{\infty} n^{\frac{4}{3}} \langle x, e_n \rangle e_n,$$

with domain

$$D((-A)^{\frac{2}{3}}) = \{x \in X, \sum_{n=1}^{\infty} n^{\frac{4}{3}} \langle x, e_n \rangle e_n \in X\}.$$

It is known that A generates a compact analytic semigroup $\{S(t)\}_{t \geq 0}$ in X , and is given by (see [21])

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, e_n \rangle e_n,$$

for $x \in X$ and $t \geq 0$. Since the semigroup $\{S(t)\}_{t \geq 0}$ is analytic, there exists a constant $M > 0$ such that $\|S(t)\|^2 \leq M$ for every $t \geq 0$. In other words, the condition $(\mathcal{H}.1)$ holds.

If we choose $\alpha \in (\frac{3}{4}, 1)$,

$$S_{\alpha}(t)x = \int_0^{\infty} \alpha \theta \eta_{\alpha}(\theta) S(\theta t^{\alpha}) d\theta, \quad x \in X.$$

Further, the operator $B : \mathbb{R} \rightarrow X$ is a bounded linear operator defined by $Bu(t)(\xi) = c(\xi)u(t)$, $0 \leq \xi \leq 1$, $c(\xi) \in L^2([0, 1])$, and the operator $W : L^2([0, T], U) \rightarrow X$ is given by

$$Wu(\xi) = \int_0^T (T-s)^{\alpha-1} S_{\alpha}(T-s) c(\xi) u(s) ds, \quad 0 \leq \xi \leq 1,$$

W is linear and by Hölder inequality, we can show that W is bounded operator but not necessarily one-to-one. Let

$$\text{Ker } W = \{x \in L^2([0, T], U), Wx = 0\}$$

be the null space of W and $[\text{Ker } W]^{\perp}$ be its orthogonal complement in $L^2([0, T], U)$. Let $\widetilde{W} : [\text{Ker } W]^{\perp} \rightarrow \text{Range}(W)$ be the restriction of W to $[\text{Ker } W]^{\perp}$, \widetilde{W} is necessarily one-to-one operator. The inverse mapping theorem says that \widetilde{W}^{-1} is bounded since $[\text{Ker } W]^{\perp}$ and $\text{Range}(W)$ are Banach spaces. So that W^{-1} is bounded and takes values in $L^2([0, T], U) \setminus \text{Ker } W$, hypothesis $(\mathcal{H}.5)$ is satisfied.

We choose the phase function $h(s) = e^{2s}$, $s < 0$, then $l = \int_{-\infty}^0 h(s) ds = \frac{1}{2} < \infty$, and the abstract phase space \mathcal{B}_h is Banach space with the norm

$$\|\varphi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{\theta \in [s, 0]} (\mathbb{E} \|\varphi(\theta)\|^2)^{\frac{1}{2}} ds.$$

To rewrite the initial-boundary value problem (4.1) in the abstract form (1.1), we assume the following:

For $(t, \varphi) \in [0, T] \times \mathcal{B}_h$, where $\varphi(\theta)(\xi) = \varphi(\theta, \xi)$, $(\theta, \xi) \in (-\infty, 0] \times [0, 1]$, we put $v(t)(\xi) = v(t, \xi)$. Define $g : [0, T] \times \mathcal{B}_h \rightarrow X$, $f : [0, T] \times \mathcal{B}_h \rightarrow X$ by

$$(-A)^{\frac{2}{3}} g(t, \varphi)(\xi) = \int_{-\infty}^0 e^{-4\theta} \varphi(\theta)(\xi) d\theta,$$

$$f(t, \varphi)(\xi) = \int_{-\infty}^0 \mu(t, \xi, \theta) f_1(\varphi(\theta)(\xi)) d\theta,$$

where

(i) the function $\mu(t, \xi, \theta) \geq 0$ is continuous in $[0, T] \times [0, 1] \times (-\infty, 0)$,

$$\int_{-\infty}^0 \mu(t, \xi, \theta) d\theta = p_1(t, \xi) < \infty, \quad \text{and} \quad \left(\int_0^1 p_1^2(t, \xi) \right) \frac{1}{2} = p(t) < \infty;$$

- (ii) the function $f_1(\cdot)$ is continuous, $0 \leq f_1(v(\theta, \xi)) \leq \vartheta(\|v(\theta, \cdot)\|_{L^2})$ for $(\theta, \xi) \in (-\infty, 0) \times (0, 1)$, where $\vartheta(\cdot) : [0, \infty) \rightarrow (0, \infty)$ is continuous and nondecreasing.

By the similar method as in Balasubramaniyam and Ntouyas [2], we can show that the assumptions (H.2) – (H.3) are satisfied.

In order to define the operator $Q : Y := L^2([0, 1], \mathbb{R}) \rightarrow Y$, we choose a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^+$, set $Qe_n = \lambda_n e_n$, and assume that

$$\text{tr}(Q) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty.$$

Define the fractional Brownian motion in Y by

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta^H(t) e_n,$$

where $H \in (\frac{1}{2}, 1)$ and $\{\beta_n^H\}_{n \in \mathbb{N}}$ is a sequence of one-dimensional fractional Brownian motions mutually independent. Let us assume the function $\sigma : [0, +\infty) \rightarrow \mathcal{L}_2^0(L^2([0, 1]), L^2([0, 1]))$ satisfies

$$\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^{2p} ds < \infty, \quad \text{for some } p > \frac{1}{2\alpha - 1}.$$

Then all the assumptions of Theorem 3.6 are satisfied. Therefore, we conclude that the system (4.1) is controllable on $(-\infty, T]$.

REFERENCES

- [1] H.M. Ahmed. On some fractional stochastic integrodifferential equations in Hilbert spaces. International Journal of Mathematics and Mathematical 2009, 2009, DOI 10.1155/2009/568078. Article ID 568078, 8 pages.
- [2] P. Balasubramaniyam and S.K. Ntouyas. Controllability for neutral stochastic functional differential inclusion with infinite delay in abstract space. J. Math. anal appl. 324 (2006), 161-176.
- [3] F. Biagini, Y. Hu, B. Øksendal, and T. Zhang. Stochastic Calculus for Fractional Brownian Motion and Application. Springer-Verlag, (2008).
- [4] B. Boufoussi, S. Hajji, and E. Lakhel. Functional differential equations in Hilbert spaces driven by a fractional Brownian motion. Afrika Matematika, 23 (2) (2012), 173-194.
- [5] B. Boufoussi and S. Hajji. Neutral stochastic functional differential equation driven by a fractional Brownian motion in a Hilbert space, Statist. Probab. Lett., 82 (2012), 1549-1558.
- [6] T. Caraballo, M.J. Garrido-Atienza, and T. Taniguchi. The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion, Nonlinear Analysis, 74 (2011), 3671-3684.
- [7] R. Coelho and L. Decreusefond. Video correlated traffic models communications networks. In Proceedings to the ITC Seminar on Telegraphic management (1995).
- [8] J. Cui and L. Yan. Existence result for fractional neutral stochastic integrodifferential equations with infinite delay. Journal of Physics A: Mathematical and Theoretical 44 (2011), 1-16.
- [9] NT. Dung. Stochastic Volterra integro-differential equations driven by fractional Brownian motion in Hilbert space, Stochastics, 87 (1) (2015), 142-159.
- [10] MM. El-Bori. On some stochastic fractional integrodifferential equations. Advances in Dynamical Systems and Applications 1 (2006), 49-57.
- [11] AA. Kilbas, H.M. Srivastava and JJ. Trujillo. Theory and applications of fractional differential equations. Elsevier, Amsterdam (2006).
- [12] J. Klamka. Stochastic controllability of linear systems with delay in control, Bull. Pol. Acad. Sci. Tech. Sci., 55 (2007), 23-29.
- [13] J. Klamka. Controllability of dynamical systems. A survey. Bull. Pol. Acad. Sci. Tech. Sci., 61 (2013), 221-229.
- [14] J. R. León and C. Lundenä. Estimating the diffusion coefficient for diffusions driven by fBm. Stat. inference and Stoch. Process (2000).
- [15] E. Lakhel. Controllability of Neutral Stochastic Functional Integro-Differential Equations Driven By Fractional Brownian Motion. Stochastic Analysis and Applications (To appear).

-
- [16] E. Lakhel and S. Hajji. Existence and Uniqueness of Mild Solutions to Neutral SFDEs driven by a Fractional Brownian Motion with Non-Lipschitz Coefficients. *Journal of Numerical Mathematics and Stochastics*, 7 (1) (2015), 14-29.
 - [17] E. Lakhel and M. A. McKibben. Controllability of Impulsive Neutral Stochastic Functional Integro-Differential Equations Driven by Fractional Brownian Motion. Chapter 8 In book : *Brownian Motion: Elements, Dynamics, and Applications*. Editors: M. A. McKibben & M. Webster. Nova Science Publishers, New York, 2015, pp. 131-148.
 - [18] K. Li. Stochastic delay fractional evolution equations driven by fractional brownian motion. *Mathematical Methods in the Applied Sciences* 2014. DOI 10. 1002/mma. 3169.
 - [19] Y. Li and B. Liu. Existence of solution of nonlinear neutral functional differential inclusion with infinite delay. *Stoc. Anal. Appl.* 25 (2007), 397-415.
 - [20] B. Mandelbrot and V. Ness. Fractional Brownian motion, fractional noises and applications. *SIAM Reviews*, 10(4) (1986), 422-437.
 - [21] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York (1983).
 - [22] Y. Ren, X. Cheng, and R. Sakthivel. On time-dependent stochastic evolution equations driven by fractional Brownian motion in Hilbert space with finite delay. *Mathematical methods in the Applied Sciences*, 37 (2013), 2177-2184.
 - [23] Y. Ren, L. Hu, and R. Sakthivel. Controllability of impulsive neutral stochastic functional differential inclusions with infinite delay, *J. Comput. Appl. Math.*, 235 (8) (2011), 2603-2614.
 - [24] R. Sakthivel, R. Ganesh, Y. Ren, and S. M. Anthoni. Approximate controllability of nonlinear fractional dynamical systems. *Commun. Nonlinear Sci. Numer. Simul.*, 18 (2013), 3498-3508.
 - [25] R. Sakthivel, P. Revathi and Y. Ren. Existence of solutions for nonlinear fractional stochastic differential equations. *Nonlinear Anal.* 81 (2013), 70-86.
 - [26] R. Sakthivel, P. Revathi and NI. Mahmodov. asymptotic stability of fractional stochastic neutral differential equations with infinite delays. *Abstract and Applied Anal.* 2013 (2013), 1-9. Article ID 769257.
 - [27] Y. Zhou, J. Feng. Existence of mild solutions for fractional neutral evolution equations. *Comput. Math. Appl.* 59 (2010), 1063-1077.
 - [28] Y. Zhou. *Basic theory of fractional differential equations*. World Scientific Publishing Co. Pte. Ltd. (2014).
 - [29] Y. Zhou, J. Wang and M. Medved. On the solvability and optimal controls of fractional integrodifferential evolution systems with infinite delay. *J. Optim. theory App.* 152 (2012), 31-50.